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Relative locality and the soccer ball problem

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We consider the behavior of macroscopic bodies within the framework of relative locality [1], which is a recent proposal for Planck scale modifications of the relativistic dynamics of particles which are described as arising from deformations in the geometry of momentum space. These lead to the addition of non-linear terms to the energy-momentum relations and conservation laws, which are suppressed by powers of ratio between the energy E of the particles involved and the Planck mass M_P. We consider and resolve a common objection against such proposals, which is that, even if the corrections are small for elementary particles in current experiments, they are huge when applied to composite systems such as soccer balls, planets and stars, with energies E_{macro} much larger than M_P. We show that this *"soccer-ball problem"* does not arise within the framework of relative locality, because the non-linear effects for the dynamics of a composite system with N elementary particles appear at most of order $E_{macro}/N \cdot M_P$.

I. INTRODUCTION

There are general reasons to suspect the existence of a regime of quantum gravity phenomena which may manifest itself as corrections to the basic relations of special relativistic particle dynamics of order of powers of Energies$/M_P$ where the Planck mass is $M_P = \sqrt{\frac{\hbar}{G_{Newton}}}$. Over the last decade several different experiments and astrophysical observations have reached sensitivity levels suitable for testing the presence of such terms, making this regime a possible site for the first experimental discovery of quantum gravitational phenomena. In a recent paper, [1] we proposed a general framework called *relative locality* which encompasses a class of such theories, based on the notion that the deformations of special relativistic physics can be coded by curvature and other non-linear deformations of momentum space.

In these theories we expect that the energy-momentum relations of relativistic particles may receive non-linear corrections which arise from attributing to momentum space, $\mathcal{P}$, a non-trivial curved metric. The energy-momentum relations then take the general form $D^2(p) = m^2$. Here $D(p)$ is interpreted as the *distance* from p to the zero energy momenta 0 with respect to a Lorentzian metric $g^{\mu\nu}(p)\mathrm{d}p_\mu \mathrm{d}p_\nu$ on momenta space. To see how these lead to corrections to special relativity, imagine that we choose coordinates on momentum space, so that the metric can be written as the Minkowski metric plus terms of order $\frac{E}{M_P}$. Then we will have

$$D^2(p) = E^2 - \vec{p}^2 - \eta \frac{E}{M_P}\vec{p}^2 + \cdots = m^2 \tag{1}$$

with η being a numerical coefficient. We will shortly discuss how one chooses such coordinates, but for the moment let us proceed naively as this will suffice to state the problem this letter solves.

Similarly, non-linear corrections to the conservation laws of energy and momentum can be understood as arising from the choice of a non-trivial connection on momentum space, $\mathcal{P}$. As shown in [1] these arise from a new, non-linear addition rule for momentum. In the simplest case of the two-to-one process $A + B \to C$, this takes the form

$$p^{(C)}_\mu = \left(p^{(A)} \oplus p^{(B)}\right)_\mu , \tag{2}$$

We assume again the existence of a nice set of coordinates on $\mathcal{P}$ which allows us to expand this as

$$\left(p^{(A)} \oplus p^{(B)}\right)_\mu = p^{(A)}_\mu + p^{(B)}_\mu - \frac{1}{M_P}\tilde{\Gamma}_\mu{}^{\alpha\beta}\, p^{(A)}_\alpha\, p^{(B)}_\beta + \cdots \tag{3}$$

Here and in the similar formulas below $\tilde{\Gamma}$ denotes the (M_p-rescaled) connection coefficients on momentum space evaluated at the origin $p_\mu = 0$. Notice that the connection coefficients on momentum space $\Gamma_\mu{}^{\alpha\beta}(0)$ ($\equiv \tilde{\Gamma}_\mu{}^{\alpha\beta}/M_p$) have dimensions of inverse mass, so the $\tilde{\Gamma}$ are dimensionless.

In both cases we write down only the leading order corrections to the standard special relativistic expressions. This is justified in the case of elementary particles, because even for most energetic cosmic rays the ratio of their energies to M_P is of order of $10^{-8} - 10^{-9}$, and the higher order terms can be safely neglected.

II. STATEMENT OF THE SOCCER BALL PROBLEM

We can now state the *soccer ball problem* which this letter is addressed to. The laws of special relativistic dynamics are universal. They apply equally to elementary particles and to large macroscopic bodies such as planets, stars and soccer balls. Either this is the case also with relative locality or it is not the case. If it is the case that the same laws apply to soccer balls and other macroscopic bodies, then the proposal is clearly wrong, the corrections are of order $M_{\text{soccer ball}}/M_P$. Since the Planck mass is roughly of the order of $10^{-5}g$ this is a huge quantity; while we do experience to very good accuracy that macroscopical objects follows a linear addition of momenta. This is the soccer ball problem: A serious objection often raised (see, *e.g.*, Refs. [2, 3]) to dismiss any attempt to introduce non linearities in momentum space.

Relative locality, as formulated in [1], is positing that the fundamental equations of particle dynamics apply only to elementary particles. This is a departure from previous theories in which the laws of motion are universal, and is perhaps necessary to unify gravity with quantum theory. Then we still have to ask, what laws are nonetheless implied for the dynamics of soccer-balls, planets and stars? Can we derive these laws and show that there are no observable deviations from the predictions of special relativity.

To answer this query we proceed in two steps. First we will give a simple, but rather naive, answer. This serves only to highlight the key point. Then we give a more rigorous argument. The difference between the naive and the more rigorous argument, as we will see, mainly comes from being careful about the choice of coordinates on momentum space made in the course of the argument. That some choice is needed is clear, because the possibility of expanding the metric in (1) around the flat metric plus terms of order E/M_P depends on the choice of coordinates.

III. A NAIVE ARGUMENT

Here is the naive argument. Let us consider first the modified dispersion relation (1). The key observation is that the soccer ball is not an elementary system, it is composed of a huge number N of elementary particles. Let each elementary particle be described by mass shell relation (1) and assume, for simplicity, that all the particles have identical masses m and momenta p_μ. The total mass of the ball is therefore $M_{(ball)} = N\,m$ and its total momentum is $\mathbf{P}_{(ball)\mu} = N\,p_\mu$. Substituting this to (1) we easily see that

$$E^2_{(ball)} = \vec{\mathbf{P}}^2_{(ball)} + M^2_{(ball)} + \eta\,\frac{E_{(ball)}}{NM_P}\vec{\mathbf{P}}^2_{(ball)} + \dots \qquad (4)$$

Comparing (1) with (4) we see that although the deformation is still present its magnitude is governed now not by the scale M_P but by the scale N times bigger, which renders the last term in (4) negligible for all practical purposes.

Let us apply the same naive argument to the non-linear conservation law (3). Let us now assume that we deal with two macroscopic bodies, body A containing N particles with identical momenta p^A_μ and body B also containing N particles with identical momenta p^B_μ. The total momentum of body A may be, naively, *defined* to be just

$$\mathbf{P}^{(A)}_\mu = N p^A_\mu \qquad (5)$$

and likewise for $\mathbf{P}^{(B)}_\mu = N p^B_\mu$. Let us assume that the collision of the two bodies can be described as the set of events in which one particle from body A interacts according to (2) with one particle from body B. We then easily see that

$$\left(\mathbf{P}^{(A)} \oplus \mathbf{P}^{(B)}\right)_\mu = \mathbf{P}^{(A)}_\mu + \mathbf{P}^{(B)}_\mu - \frac{1}{NM_P}\tilde{\Gamma}_\mu{}^{\alpha\beta}\,\mathbf{P}^{(A)}_\alpha\,\mathbf{P}^{(B)}_\beta + \cdots \qquad (6)$$

Hence the same conclusion holds that the non-linearities are damped by powers of NM_P.

This argument is too naive on two counts. First, we need to be specific about the choice of coordinates on momentum space. Second, the definition (5) is not fully justified. We have made the other equations of relativistic dynamics non-linear; why not that one also? Shouldn't we expect that the definition of the total momenta also involves non linearity? We now address each in turn and the result is a more rigorous argument to the same conclusion.

IV. A MORE RIGOROUS ARGUMENT

A. The choice of coordinates on momentum space

As we stressed, in order to write down the formulas (1), (2), one has to choose a coordinate system on the momentum space. The coordinates must be such that the origin corresponds to the state with zero momentum and that both modified dispersion relation and momentum composition rules become the standard special relativistic ones in the limit of vanishing momentum space curvature or $M_P \to \infty$. The archetypal example of such coordinate system is provided by Riemann normal coordinates. In Riemann normal coordinates the metric geodesics from the origin are straight lines and we find

$$m^2 = D^2(p) \equiv \eta^{\mu\nu} p_\mu p_\nu\,, \qquad (7)$$

and therefore the dispersion relation in normal coordinates is not modified. In this case therefore the whole information about the momentum space curvature is contained in the deformed momentum composition rule (2). In any other coordinate system, the dispersion relation, still defined as $m^2 = D^2(p)$ would take the general form (1).

Another important coordinate which we will use is the connection normal coordinates, for which the geodesics associates with the connection are straight lines, even if the connection is not metrical. In these coordinates $\hat{p}$ the addition of parallel momenta is linear i.e.

$$(a\hat{p})\hat{\oplus}(b\hat{p}) = (a+b)\hat{p} \qquad (8)$$

where a, b are any scalars. At first order in M_P the connection coordinates are given by $\hat{p}_\mu = F_\mu(p)$ where

$$F_\mu(p) = p_\mu + \frac{1}{2M_P}\tilde{\Gamma}^{\alpha\beta}_\mu p_\alpha p_\beta + \cdots \qquad (9)$$

The addition in the new coordinates is given by $\hat{p}\hat{\oplus}\hat{q} \equiv F(F^{-1}(\hat{p}) \oplus F^{-1}(q))$ while its expansion is

$$\hat{p}\hat{\oplus}\hat{q} = \hat{p}_\mu + \hat{q}_\mu - \frac{1}{M_P}\,\tilde{\Gamma}_\mu{}^{[\alpha\beta]}\,\hat{p}_\alpha\,\hat{q}_\beta + \cdots \qquad (10)$$

where the bracket denotes antisymmetrization. Since only the torsion component at $p_\mu = 0$ enters at first order we obtain the desired result. If the connection is metrical, that is if $\nabla^\mu g^{\alpha\beta} = 0$ then the Riemann and connection normal coordinates agree. If the connection is non metrical the metric geodesics and connection geodesics no longer agree. We note that this would have interesting phenomenological consequences [4].

B. A model of macroscopic bodies in collision

Now that we have discussed the choice of coordinates we are ready to present a careful analysis of the soccer ball problem. This involves an idealization of the properties of macroscopic body, then we will make it slightly less idealized.

We first consider an idealized situation, involving two bodies "A" and "B" each composed of N atoms. Let us assume that in the course of their interaction the bodies exchange photons. Denoting the photon's momentum by k_μ and the initial and final momentum of the atom by p_μ and $\tilde{p}_\mu$, respectively we find that for the photon emission process we have

$$p_\mu = (\tilde{p} \oplus k)_\mu \tag{11}$$

while for photon absorption

$$(k \oplus p)_\mu = \tilde{p}_\mu \tag{12}$$

Let us now consider the process of a single photon exchange between the body A and B. Assuming that the body A emits the photon, and body B absorbs it we find the relations

$$p^A_\mu = (\tilde{p}^A \oplus k)_\mu\,, \quad (k \oplus p^B)_\mu = \tilde{p}^B_\mu\,.$$

Solving these two equations for k we find the relation.

$$[(\ominus \tilde{p}^A \oplus p^A) \oplus p^B]_\mu = \tilde{p}^B_\mu \tag{13}$$

where we introduced the antipode of momentum $\ominus p$ defined by $(\ominus p) \oplus p = 0$ and used the left inverse property $(\ominus p) \oplus (p \oplus q) = q$. In the leading order the antipode is given by

$$(\ominus p)_\mu = -p_\mu - \frac{1}{M_P}\,\Gamma_\mu{}^{\alpha\beta}\,p_\alpha\,p_\beta + \ldots$$

Eq. (13) describes the momentum conservation rule of a single interaction (emission and absorption) process. The soccer ball problem would arise if the same form of the conservation rule would hold for macroscopic, massive bodies with initial and final total momenta $P^{A,B}_\mu$ and $\tilde{P}^{A,B}_\mu$, respectively, i.e. if we had

$$[(\ominus \tilde{P}^A \oplus P^A) \oplus P^B]_\mu = \tilde{P}^B_\mu\,. \tag{14}$$

Let us now show that this naive expectation is not fulfilled in the case of a large body composed of a large number of microscopic subsystems. To see this let us assume that in the interaction process each of the N atoms of the body A emits one and only one photon that is subsequently absorbed by one and only one atoms of the body B. For each such process we have to do with the conservation rule (13) so that for each interacting pair of constituents of the bodies A and B, labeled by index a, $a = 1,\ldots,N$ we can write

$$[(\ominus \tilde{p}^{A_a} \oplus p^{A_a}) \oplus p^{B_a}]_\mu = \tilde{p}^{B_a}{}_\mu \tag{15}$$

Expanding this expression to the leading order in $1/M_P$ we get the relation

$$\begin{aligned}&[p^{A_a} + p^{B_a}]_\mu - \frac{1}{M_P}\tilde{\Gamma}^{\alpha\beta}_\mu p^{A_a}_\alpha p^{B_a}_\beta + \cdots \\ =&[\tilde{p}^{A_a} + \tilde{p}^{B_a}]_\mu - \frac{1}{M_P}\tilde{\Gamma}^{\alpha\beta}_\mu \tilde{p}^{A_a}_\alpha \tilde{p}^{B_a}_\beta + \cdots\end{aligned} \tag{16}$$

Note that to leading order the LHS of this equation is just $p^{A_a} \oplus p^{B_a}$ and the equation expresses that this is equal to this order to $\tilde{p}^{A_a} \oplus \tilde{p}^{B_a}$. This is a momentum conservation equation which expresses that the non linear addition is the one preserved in interactions mediated by photons exchange.

C. The definition of the total momentum of a body

Let us now define the macroscopic momentum to be the non linear composition of the microscopic momenta in some order

$$\mathbf{P}^A \equiv p^{A_1} \oplus (p^{A_2} \oplus (\cdots \oplus p^{A_N})\cdots) \tag{17}$$

with similar expressions for the total momentum of the other body. Involving the non linear addition in the definition of the total momenta is motivated by the last remark in the previous section. In addition, let us assume for simplicity that all the momenta of microscopic constituents are identical, to wit $\forall_a\, p^{A_a}_\mu = p^A_\mu$ etc. This is a simplistic model of a macroscopic body, of course, but it makes it possible to capture the relevant features of interacting macroscopical bodies at play in the soccer ball issue. Then, once we chose to work in the connection normal coordinates, we can use the fact that the non linear addition of colinear momenta is equal to the ordinary linear addition. Hence, in these coordinates we have that remarkably, $\mathbf{P}^A_\mu$ equals just $N p^A_\mu$. We can then easily sum up expressions (16) over a to obtain

$$[\mathbf{P}^A + \mathbf{P}^B]_\mu - [\tilde{\mathbf{P}}^A + \tilde{\mathbf{P}}^B]_\mu = \frac{\tilde{\Gamma}^{[\alpha\beta]}_\mu}{N\,M_P}\left(\mathbf{P}^A_\alpha \mathbf{P}^B_\beta - \tilde{\mathbf{P}}^A_\alpha \tilde{\mathbf{P}}^B_\beta\right) \tag{18}$$

We thus arrive at the same conclusion as the naive argument (6), that is the non linerarities for macroscopical bodies are damped by powers of NM_P nstead of M_P, but this time on firm ground.

We see therefore that in the limit of large number of elementary constituents N the linearly combined momenta satisfy with good accuracy the standard, linear conservation law. Indeed, if the elementary microscopic constituents were atoms the ratio of the second and first terms is of order of 10^{-18}. Thus the soccer ball problem is avoided. This is the main result of this note.

D. Making the model slightly less idealized

We can now also consider the case where the individual atomic momenta fluctuate around the mean value p thus

$$p^{A_a} = p^A + \delta p^{A_a} \tag{19}$$

where δp^{A_a} are small fluctuation that average to zero in time and when we sum over a. We note that these fluctuations have to be small if the constituents cohere into a macroscopic body, as assumed. First one sees, using the property of the connection normal coordinates, and the fact that $\sum_a \delta p^{A_a} = 0$, that the non linear addition (17) differ from the linear one by a term

equal to $\frac{1}{M_P}\tilde{\Gamma}_\mu^{[\alpha\beta]}\sum_{a<b}\delta p_\alpha^{A_a}\delta p_\beta^{A_b}$. For a macroscopic body, i.e. for large N, one can safely estimate this term by examining the average value of $\tilde{\Gamma}_\mu^{[\alpha\beta]}\delta p_\alpha^{Aa}\delta p_\alpha^{Ab}$, also exploiting the fact that $\tilde{\Gamma}_\mu^{[\alpha\beta]}$ is antisymmetric. It is natural to assume that for large N

$$\left\langle \delta p_\alpha^{A_a}\delta p_\beta^{A_b}\right\rangle \sim \frac{P^2}{N^2}\left(\delta^{ab}-\frac{v^a v^b}{N}\right)\sigma_{\alpha\beta} \qquad (20)$$

where v^a is a vector with all components equal one, such that $v^a\delta p_a = 0$. The average $\langle\cdot\rangle$ taken here, denotes an average over time, but a similar result is obtained for an ensemble average. In light of eq. (20) we can safely estimate that the correction term is smaller than $\tilde{\Gamma}PP/N^3M_P$ and therefore negligible or even vanishing since the fluctuation tensor $\sigma_{\alpha\beta}$ is symmetric. These fluctuations also enter the non linear conservation (16) adding terms proportional to $\tilde{\Gamma}_\mu^{\alpha\beta}\sum_a\langle\delta p_\alpha^{A_a}\delta p_\beta^{B_a}\rangle$ whose average vanish since the fluctuation of the body A and B are decorrelated.

V. CONCLUSIONS

We have seen that the so called "soccer ball problem" is not present in the relative-locality framework because the total momentum of a macroscopic body is equal, within a very small margin, to the linear sum of momenta of the constituents, i.e.

$$\mathbf{P}^{total} = \sum_a p^{A_a}. \qquad (21)$$

We established this rigorously exploiting a choice of momentum-space coordinates for which this addition is linear for colinear momenta. But we could turn around the reasoning and use our argument to show that the quantity (21) is macroscopically conserved in the interaction between two macroscopic bodies. This suggest that (21) can serve as the definition of what the momentum of a large composite system is. That is we could define the total momenta as a quantity which is conserved in the scattering processes provided it is also preserved by the internal dynamics that bound the constituent of the macroscopical body together.

Notice also that the number N can be interpreted not only as a number of elementary particles of the bodies, but as being proportional to the number of elementary interactions. In the realistic situation of scattering of two macroscopic bodies which are approximately rigid not all the elementary constituents of body A interact with those of the body B. However, each time a photon is emitted and absorbed by a constituent of the body the total momentum transfer is then quickly redistributed by the internal interactions insuring the rigidity of the body to all the constituents of the body This redistribution happens if the initial momentum transfer do not excite phonons interactions and can be assume to happen over a time shorter than the total interaction time between the two macroscopical bodies. This process involves at least as many interactions as there are particles in the body which makes the estimate (18) valid even in this, more realistic case.

We note that the idea that the soccer ball problem is solved because the non-linearities relevant for the dynamics of a system composed of N elementary particles are suppressed by a mass scale NM_P is not new. It has been proposed a number of times before (see *e.g.* Refs. [5–7]). However, here we have shown it to be the case in a well defined class of theories with curved momentum spaces.

We should also mention that there is an aspect of the soccer ball problem that deserved to be studied under the same lines developped here, which is to check consistency of the transformation properties under boosts of the fundamental particles with the usual transformation properties of the total momenta.

To conclude, we see that the soccer ball problem does not occur in theories satisfying the Principle of Relative Locality. In fact, as stressed some time ago in [8] the real question is how to find a system which would exhibit large enough deformations to be detectable in a feasible experimental setup.

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